Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
Second Semester - Analysis IV

Midterm Exam — Date: February 27, 2018
Maximum marks: 40 — Duration: 3 hours

Answer any four, each question carries 10 marks, total marks: 40

1. (a) Let $\mathcal{A}$ be an algebra of complex-valued continuous functions on a compact metric space X that separates points of X and nowhere vanishes on X. If $\mathcal{A}$ is self-adjoint, prove that $\mathcal{A}$ is dense.

   (b) Prove that $C[0,1]$ has no open set whose closure is compact (**Marks: 5**).

2. Let E be a set of continuous functions on a compact metric space. Prove that $\overline{E}$ is compact if and only if E is equicontinuous and pointwise bounded.

3. Let f be a continuously differentiable 2π–periodic function and s_n be the n-th partial sum of the Fourier series of f.

   (a) Prove that $s_n \to f$ uniformly (**Marks: 5**).

   (b) Further if $\int_{-\pi}^{\pi} f(x)dx = 0$, prove that $||f'|| \geq ||f||$ and the equality occurs if and only if $f(x) = acosx + bsinx$ where $||f||^2 = \int_{-\pi}^{\pi} |f|^2, ||f'||^2 = \int_{-\pi}^{\pi} |f'|^2$.

4. (a) Prove Riemann-Lebesgue Lemma.

   (b) Prove using Fourier series that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ (**Marks: 5**)?

5. (a) Let $f \in \mathcal{R}[-\pi,\pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series of f at $x \in \mathbb{R}$. Prove that

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}dt.$$

   (b) Let $f \in \mathcal{R}[-\pi,\pi]$ be a 2π-periodic function such that $f(x) = \cos x$ for all $x \in [0,\pi]$. Discuss the convergence of $s_n(x)$ for all $x \in (0,\pi)$ (**Marks: 4**).

6. (a) Prove that the Fourier series of a bounded 2π-periodic function that is monotonic on $[-\pi,\pi)$ converges.

   (b) Let ϕ be a step function and $\phi(x) \sim a_0/2 + \sum_{n=1}^\infty a_n \cos nx + b_n \sin nx$. Prove that there is a constant C such that $|a_n| \leq C/n$ for all $n \geq 1$ (**Marks: 4**).